Certain representations of the wreath product and of a certain type of its subgroups

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1. Introduction

Let A and B be two groups. Then the (abstract) wreath product $A \wr B$ of A and B is one way of defining new groups from A and B[1]. If A and B are permutation groups then a permutation group $A \wr B$ can be defined ([5], [4] p. 81). This permutation group is isomorphic to the abstract group $A \wr B$ if and only if B is a regular permutation group. We generalise in what follows the abstract definition of $A \wr B$ in the sense that the group structures of the permutation groups $A \wr B$ can be computed from generators and defining relations as well.
Furthermore, representations of such general wreath products are considered (B is here supposed to be finite). The discussion is carried through in terms of modules: starting from a module M over the group algebra KA of the group A over a field K, and a transitive permutation representation of B, a $KA \wr B$-module W is constructed. A sufficient condition for the irreducibility of W is derived (corollary 1 to theorem 2) as a special case of a general irreducibility condition for modules W_G, where G is a subgroup of a certain type of $A \wr B$ (see section 3). It appears that W is irreducible if the matrix representation of A afforded by M does not consist of matrices all of which have 1 as a characteristic value. This condition is not necessary as has been shown by considering a special case (theorem 3). The irreducibility condition for W_G however is pointed out to be also a necessary condition if we take for G metacyclic groups while K is algebraically closed with characteristics not dividing the order of G. The $KA \wr B$-modules play a

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role in the definition of generalized transfer maps in the cohomology of groups [3].

2. Definition of $A \wr B$ and of the modules W

Let A and B be two groups; then $A \wr B$ as an abstract group is defined as follows. Let for every $b \in B$, A_b be a copy of A. Let ΠA_b be the (restricted resp. unrestricted) direct product of the A_b; then the (restricted resp. unrestricted) wreath product $A \wr B$ in the group generated by the (restricted resp. unrestricted) product group ΠA_b, together with the elements of B, where multiplication of the elements of ΠA_b with the elements of B is defined by the relations $b^{-1} a_{b_1} b = a_{b_1 b}$, for $a_b \in A_b$, $b, b_1 \in B$. $A \wr B$ contains ΠA_b as an invariant subgroup with factor group B. It is a splitting subgroup. We assume in what follows B to be of finite order, so that the distinction between restricted and unrestricted wreath product will not play a role.

Lemma 1. Let $\phi\colon A \wr B \to F$ be a group homomorphism of $A \wr B$ onto the group F. Let ϕ be such that the image of ΠA_b under ϕ has the form $\Pi' \phi(A_b)$, where Π' means that the product is taken by letting b run through some subset of B. We have for all $b', b \in B$ that $\phi(A_b) \cong \phi(A_{b'})$. Let moreover b_1 be a fixed chosen element of B, then the set

$$B' = \{b \mid b \in B, \forall a \in A\colon \phi(b^{-1})\phi(a_{b_1})\phi(b) = \phi(a_{b_1})\}$$

is a subgroup of B. B' consists precisely of those elements $b \in B$ for which the equality $\phi(A_b) = \phi(A_{b_1})$ holds. The right coset $B'b_o$ $(b_o \in B)$ of B' consists of precisely those elements $b \in B$ for which $\phi(A_b) = \phi(A_{b_1 b_o})$ holds.

Proof. Using $\phi(b^{-1})\phi(a_{b_o})\phi(b) = \phi(a_{b_o b})$, which holds true for all $b, b_o \in B$ and all $a \in A$, we find that $\phi(A_b) \cong \phi(A_{b_o})$. It follows from same relations that B' is a subgroup of B. Now, if $b \in B'$, then let $b_o \in B'$ be such that $b = b_1 b_o$. Then $\phi(b_o^{-1})\phi(a_{b_1})\phi(b_o) = \phi(a_{b_1 b_o}) = \phi(a_{b_1})$

for all $a \in A$, which means $\phi(A_b) = \phi(A_{b_1})$. Let, inversely, $\phi(A_b) = \phi(A_{b_1})$, then $\phi(b)\phi(a_b)\phi(b^{-1}) = \phi(b)\phi(a_{b_1})\phi(b^{-1}_1) = \phi(a_1)$ for all $a \in A$. But $\phi(A_1) = \phi(A_{b_1})$ from which $b^{-1}_1 \in B'$ and $b \in B'$ follow. The last proposition of the lemma follows by same kind of reasoning.

We are in what follows interested in group homomorphisms ϕ of the kind as described in lemma 1, and try to compose homomorphisms of that kind. From lemma 1 we see that $\Pi'\phi(A_b)$ is a splitting normal subgroup of F with factor group $\phi(B)$. The product $\Pi'\phi(A_b)$ appears to be extended over a set of different coset representatives of a certain subgroup B' of B, while the set of factors $\phi(A_b)$ is permuted transitively by premultiplication by $\phi(b^{-1})$ and postmultiplication by $\phi(b)$. We therefore start out with a given subgroup $B' \subset B$. B' defines a transitive permutation representation of B, which is given by those permutations of the right cosets $B'b$ ($b \in B$) of B' which are induced from the right regular representation $x \to xb$ ($b \in B$) of B (see [4], p. 57).

Now, let $\phi: A \to A'$ be a homomorphism of the group A onto the group $A' = \phi(A)$. Let $\phi_x: A_x \to A'_x$ denote same homomorphism ϕ except that it is applied upon a copy A_x of A and ends up in a copy $A'_x = \phi_x(A_x)$ of A'. In order to avoid excessive notation we denote in what follows the group $\phi_x(A_x)$ by $\phi(A_x)$. Let for every B'b the symbol $A_{B'b}$ denote a copy of A. Then let for every $b \in B$, ϕ_b be a homomorphism of A_b onto $A'_{B'b} = \phi(A_{B'b})$ defined by the map $a_b \to \phi(a_{B'b})$ ($a \in A$). We try to extend the homomorphisms ϕ_b to a homomorphism of ΠA_b onto $\Pi\,\phi(A_{B'b})$. This is not necessarily possible for every choice of subgroup $B' \subset B$.

<u>Lemma 2</u>. Given a subgroup B' of B, the homomorphisms $\phi_b: A_b \to \phi(A_{B'b})$ defined by the mappings $a_b \to \phi(a_{B'b})$ ($a \in A$) can be extended to a homomorphism of ΠA_b onto $\Pi\,\phi(A_{B'b})$ if and only if one of the following conditions is fulfilled:

(i) B' is any subgroup of B and $\phi(A)$ is abelian,

(ii) B' is the trivial subgroup of B and $\phi(A)$ is non-abelian.

If such an extension homomorphism exists, then it is unique.

Proof. It is sufficient to prove that ϕ_b can be extended to a homomorphism of $\prod_{b \in B'} A_b$ into $\phi(A_{B'})$. Such an extension homomorphism necessarily maps an element $(a_b) \in \prod_{b \in B'} A_b$ onto the product in $\phi(A_{B'})$ of the images $\phi(a_b)$ of the components of (a_b). We see readily that if $\phi(A_{B'})$ is abelian then such an extension exists and is unique, whatever $B' \subset B$ is taken. If on the other hand $\phi(A_{B'})$ is non-abelian and if $B' \neq \{1\}$, then one finds easily an element in $\prod_{b \in B'} A_b$ that is carried into two different elements of $\phi(A_{B'})$, under the correspondence described above.

Proposition 1. Let a group homomorphism $\phi: A \to A' = \phi(A)$ be given. Let furthermore $B' \subset B$ be a subgroup of B such that the derived homomorphisms ϕ_b of lemma 2 are extendable to a homomorphism (called also ϕ) of $\prod A_b$ onto $\prod \phi(A_{B'b})$. Denote the permutation representation induced from the right regular representation of B on B' by $\pi(B)$. Then the group $H(\phi(A), \pi(B))$ generated by the groups $\prod \phi(A_{B'b})$ and $\pi(B)$ with defining relations $\pi(b_o^{-1})\phi(a_{B'b})\pi(b_o) = \phi(a_{B'bb_o})$ $(b_o, b \in B)$ is homomorphic to $A \wr B$ under the map

$$(\phi, \pi): b(a_b) \to \pi(b)(\phi(a_{B'b})) \quad ,$$

where $(\phi(a_{B'b}))$ is the image in $\prod \phi(A_{B'b})$ of (a_b) in $\prod A_b$ under ϕ.

Proof. The proposition can be verified immediately.

Remark. The definition of $H(\phi(A), \pi(B))$ is in fact independent from the extendability of the ϕ_b and from the finiteness of B.

Let $P(\phi(A))$ be an arbitrary isomorphic permutation representation of $\phi(A)$, then $H(\phi(A), \pi(B))$ is as an abstract group isomorphic to the wreath product $P(\phi(A)) \wr \pi(B)$ of the permutation groups $P(\phi(A))$ and $\pi(B)$. Those permutation groups $P(\phi(A)) \wr \pi(B)$ were first introduced in [5].

The modules W are obtained as follows. Let $B' \subset B$ be a subgroup of B and let $\phi: A \to A' = \phi(A)$ be a given homomorphism. Let furthermore M be a finitely generated $K\phi(A)$-module with basis $\{e_i\}$. Let for every coset $B'b$, $M_{B'b} \cong M$ be a $K\phi(A_{B'b})$-module, where for every $B'b$, $\phi(A_{B'b})$ denotes a copy of $\phi(A)$ acting in same way upon a basis $\{e_i^{B'b}\}$ of $M_{B'b}$ as $\phi(A)$ acts upon the basis $\{e_i\}$ of M.

Define $W = \sum \oplus M_{B'b}$, where the summation is taken over the (different) cosets of B'. Then W becomes in the obvious way a $K\Pi\phi(A_{B'b})$-module. Let $\pi(B)$ be the permutation representation of B defined by B'. We make W into a $K\pi(B)$-module by letting the elements of $\pi(B)$ act upon the basis elements $e_i^{B'b}$ of W as follows:

$$\Pi(b_o)\, e_i^{(B'b)} = e_i^{(B'bb_o^{-1})} \qquad (b_o \in B).$$

Then we have

<u>Theorem 1</u>. Let R be the smallest ring of endomorphisms of W containing $K\Pi\phi(A_{B'b})$ and $K\pi(B)$. Then $R = KH(\phi(A),\pi(B))$, where $H(\phi(A),\pi(B))$ is the group defined in proposition 1. If $B' \subset B$ is chosen such that the homomorphisms ϕ_b of lemma 2 can be extended to ΠA_b, then by letting the elements $b'(a_b)$ of $A \wr B$ act on W in same way as their images $(\phi,\pi)\, b'(a_b) = \pi(b')(\phi(a_{B'b}))$ in $H(\phi(A),\pi(B))$ do, W becomes a $KA \wr B$-module.

<u>Proof</u>. It is sufficient to prove that for all $b \in B$, $a \in A$ and every coset $B'b^*$ the relations $\pi(b_o^{-1})\phi(a_{B'b^*})\pi(b_o) = \phi(a_{B'b^*b_o})$ between the automorphisms in $\pi(B)$ and $\Pi\phi(A_{B'b})$ of W hold true.

Let $W = \sum_{(i)} \sum_{B'b} \lambda_i^{(B'b)} e_i^{(B'b)}$ be an arbitrary vector in W. Then

$$\pi(b_o^{-1})\phi(a_{B'b^*})\pi(b_o)\, W = \pi(b_o^{-1})\phi(a_{B'b^*}) \sum_{(i)} \sum_{B'b} \lambda_i^{(B'b)} e_i^{(B'bb_o^{-1})} =$$

$$= \pi(b_o^{-1}) \sum_{(i)} \sum_{B'b \neq B'b^*} \lambda_i^{(B'b)} e_i^{(B'bb_o^{-1})} + \sum_{(i)} \lambda_i^{(B'b^*)} \phi(a_{B'b^*}) e_i^{(B'b^*)} =$$

$$= \sum_{(i)} \sum_{B'b \neq B'b^*} \lambda_i^{(B'b)} e_i^{(B'b)} + \sum_{(i)} \lambda_i^{(B'b^*)} \phi(a_{B'b^*}) e_i^{(B'b^* b_o)} =$$

$$= \phi(a_{B'b^* b_o}) W .$$

q.e.d.

Remark. The symbol W denotes in what follows a module W of the type constructed above, the operator ring being $KA \wr B$, where the elements of $A \wr B$ are defined to act upon W as defined in theorem 1. In case however that $B' \subset B$ is such that the ϕ_b cannot be extended to ΠA_b , we may consider the module W to be a $KH(\phi(A), \pi(B))$-module, or (remark on proposition 1) as a $KP(\phi(A)) \wr \pi(B)$ - module. All definitions and propositions pertaining $A \wr B$ and the module W that will be derived in what follows can be carried over to the groups $P(\phi(A)) \wr \pi(B)$ - considered as abstract groups - and the module W.

3. The class C and the modules W_G.

The letter G denotes in what follows a subgroup of $A \wr B$ with the properties:

(i) G contains a subgroup $\bar{A}$ which is a subdirect product of ΠA_b ;

(ii) a set of right coset representatives of G with respect to A is also a set of representatives of $A \wr B$ with respect to ΠA_b.

The class of groups defined by (i) and (ii) is denoted by C. Example of groups in C are $A \wr B$ itself, the group extensions G of A by B: $G/A \cong B$, which are embedded in $A \wr B$ ([5], [6]). Other groups of C are defined in [6]. Finally, we would mention the groups G with subgroup A, such that the permutation group defined by A in G is isomorphic to B (Frobenius embedding).

Let W be a $KA \wr B$ - module as defined in section 2, then W_G will denote the same module except that the operators are restricted to KG in $KA \wr B$.

Proposition 2. If $B' = \{1\}$ then W_G is isomorphic to the induced module $M_{B'}^{G}$, where $M_{B'}$ is the $K\bar{A}$-submodule of $W_{\bar{A}}$ obtained from the $K\Pi A_b$-module $M_{B'}$ by restriction of the operators to $K\bar{A}$.

Proof. $B' = \{1\}$ implies $B \cong \pi(B)$ under the map $b \to \pi(b)$. If $\pi(b)\bar{a}_b \in G$ with $\bar{a}_b \in \bar{A}$ are the representatives of $\bar{A}$ in G then we have $W_G = \sum \oplus \pi(b)\bar{a}_b M_{B'} = \sum \oplus \pi(b) M_{B'}$. The map $\sum b\bar{a}_b \otimes m_b \to \sum \pi(b)\bar{a}_b m_b$ ($m_b \in M_{B'}$) defines a KG-isomorphism of $M_{B'}^{G} = \sum b\bar{a}_b \otimes M_{B'}$ onto W_G (see [2], p. 74, 323).

Proposition 3. W_G is reducible if the $K\bar{A}$-module $M_{B'}$ is reducible.

Proof. Let $M_{B'}^{*}$ be an irreducible $K\bar{A}$-submodule of $M_{B'}$, then $\sum \oplus M_{B'b}^{*}$ is a KG-submodule of W_G.

Lemma 3. W_G is irreducible if the $K\bar{A}$-module $W_{\bar{A}}$ contains no other $K\bar{A}$-submodules than thosewhich are direct sums of $M_{B'b}$'s.

Proof. The condition of the lemma implies that the submodules of W_G must also be direct sums of $M_{B'b}$'s. A direct sum of $M_{B'b}$'s can never be a proper submodule of W_G , as the automorphisms $\pi(b)\bar{a}_b$ act transitively upon the modules $M_{B'b}$.

Let a vector $v = \sum \lambda_i e_i^{B'b} + \sum \mu_i e_i^{B'b^*}$ $(\lambda_i, \mu_i \in K)$ of the $K\bar{A}$-submodule $M_{B'b} \oplus M_{B'b^*}$ $(B'b \neq B'b^*)$ be denoted by (λ_i, μ_i). Let moreover (a_{ij}) and (a_{ij}^{*}) denote the matrix blocks in $T(a)$ $(a \in \bar{A})$, corresponding to the modules $M_{B'b}$ and $M_{B'b^*}$ respectively, in the (direct sum) matrix representation T of $\bar{A}$ afforded by $W_{\bar{A}}$. Let those blocks have degree n. Then we prove

Lemma 4. If the KA-module M is irreducible and if for all pairs of different cosets $B'b, B'b^*$ the $K\bar{A}$-module $M_{B'b} \oplus M_{B'b^*} \subset W_{\bar{A}}$ does not contain a subdirect submodule S such that there exist vectors (x_i, y_i) and (λ_i, μ_i) in S, and an $a \in \bar{A}$ with the property

$$\sum_{j=1}^{n} a^*_{ij}\, y_j \neq \mu_i \qquad (i=1,\dots,n)$$

$$\sum_{j=1}^{n} a_{ij}\, x_j = \lambda_i \qquad (i=1,\dots,n)\ ,$$

then $W_{\bar{A}}$ contains no other $K\bar{A}$-submodules than those which are arbitrary direct sums of $M_{B'b}$'s.

Proof. From the irreducibility of M and the subdirectness of $\bar{A}$ in ΠA_b it follows immediately that every $M_{B'b}$ is an irreducible submodule of $W_{\bar{A}}$. Let V be a submodule of $W_{\bar{A}}$, not equal to a direct sum of $M_{B'b}$'s. Then $M_{B'b} \subset V$ implies that $M_{B'b}$ is a direct summand of V. If we leave away all those direct summands from V, then we are left with a submodule V^* of $W_{\bar{A}}$, V^* being a submodule of the direct sum of a number of $M_{B'b}$'s, which contains no $M_{B'b}$ as a submodule.

Let S^* be the smallest non-trivial submodule of V^* in a composition series of V^*. Then it follows from the Jordan-Hölder theorem that S^* is $K\bar{A}$-isomorphic to some module $M_{B'b}$. As S^* is not equal to any $M_{B'b}$, there must exist at least two different cosets B'b and B'b* such that the $K\bar{A}$-projection S of S^* into $M_{B'b} \oplus M_{M'b^*}$ is a subdirect module of $M_{B'b} \oplus M_{B'b^*} \subset W_{\bar{A}}$ and is not equal to $M_{B'b} \oplus M_{B'b^*}$. S is an irreducible module, as S^* is irreducible. The progection $\pi(S)$ of S into $M_{B'b}$ is for that reason a $K\bar{A}$-isomorphism.

Now, let (λ_i, μ_i) and (x_i, y_i) be vectors in S such that the conditions of the lemma hold true. This means that there exist two different vectors in S, viz. (λ_i, μ_i) and $(\sum a_{ij}x_j\ ,\ \sum a^*_{ij}y_j) = (\lambda_i, \sum a^*_{ij}y_j)$, having same projection in $\pi(S)$, viz. $(\lambda_i, 0)$. This is however impossible on account of the isomorphism between S and $M_{B'b}$.

Theorem 2. The module W_G is irreducible if the conditions of lemma 4 with respect to the $K\bar{A}$-module $W_{\bar{A}}$ are satisfied.

Proof. Lemma 3 and lemma 4.

Corollary 1. Let $G = A \wr B$, then $W_{A \wr B} = W$ is irreducible if not all the matrices of the irreducible matrix representation of A afforded by M have a characteristic value equal to 1.

Proof. We have $\bar{A} = \Pi A_b$. Let S be an irreducible subdirect $K\bar{A}$-submodule of $M_{B'b} \oplus M_{B'b^*}$, then take for $(\lambda_i, \mu_i) \in S$ a vector $(1, \mu_i)$. Such a vector exists in S as the projection of S in $M_{B'b}$ is onto. Take $(x_i, y_i) = (1, \mu_i)$, $(a_{ij}) = I$ (identity matrix), $(a^*_{ij}) \neq I$. We have that $\sum a^*_{ij} \mu_j = \mu_j$ only if (a^*_{ij}) has a characteristic value equal to 1.

Corollary 2. If the degree of the irreducible representation of A afforded by M is equal to 1, then theorem 2 gives:
W_G is irreducible if for every pair of different cosets B'b and $B'b^*$ there exists an element $a \in \bar{A}$ such that for the entries α and α^* $(\alpha, \alpha^* \in K)$ corresponding to the modules $M_{B'b}$ and $M_{B'b^*}$ in the diagonal representation of $\bar{A}$ afforded by $W_{\bar{A}}$, the inequality $\alpha \neq \alpha^*$ holds.

Remark. The question whether the condition of theorem 2 is also a necessary condition for the irreducibility of W_G has to be answered in the negative (see theorem 3 below). This condition is however necessary in the following case. Let K be algebraically closed, and let A and B be both finite cyclic with order n and m, respectively. Assume that nm is not divisible by the characteristic of K. Then take for G the metacyclic groups $G/A \cong B$. The module W_G is in this case isomorphic to the induced module $M^G_{B'}$ (proposition 2). A simple calculation shows that in this case the condition of corollary 2 is equivalent to (the sufficient part of) the irreducibility criterion for $M^G_{B'}$, as has been derived in [2], §47. This condition however is also necessary (loc.cit.). It is likely that the condition of corollary 2 is also necessary if we take for G metabelian groups (A and B abelian, $G/A \cong B$).

Theorem 3. Let A and B be finite, K algebraically closed. Assume that the characteristic of K does not divide the order of $A \wr B$. Let $B' = \{1\}$. Then W is an irreducible $KA \wr B$- module if M is an irreducible KA-module.

Proof. We have $W = M_1^{A \wr B}$ (proposition 2). The assumptions of the theorem permit us to apply an irreducibility criterion for induced modules ([2], §45). According to this theorem, we have only to show that for every $b \in B$ the irreducible $K\Pi A_b$-modules M_b are not $K\Pi A_b$-isomorphic. This follows however immediately from the fact that every $A_{b'}$ acts trivially on M_b if and only if $b \neq b'$.

The only if part of the theorem follows from proposition 3.

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